Jan 17, 2014

We've been looking at tangent lines:

function $f(x)$
Slope of tangent line to $f(x)$ at $x=a$ is:

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

* what if you find the tangent line at $x=1$, but now want it at $x=2$? You would have to calculate all over again!

We can think about the slope of the tangent line at a pt x as a function:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

x is a variable
we call this the <u>derivative</u>

<u>DEF</u>: The <u>derivative</u> of a function f is a new function defined by:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

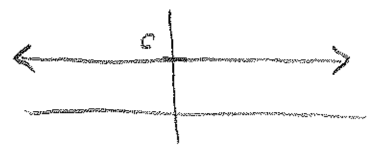
The domain of $f'(x)$ is where this limit exists.
We say f is <u>differentiable at a</u> if $f'(x)$ exists at $x=a$.

(We need the limit to exist)

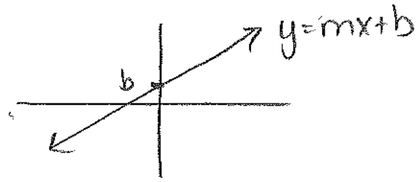
just think of this as the slope of the tangent line at each pt

<u>Some basics:</u>

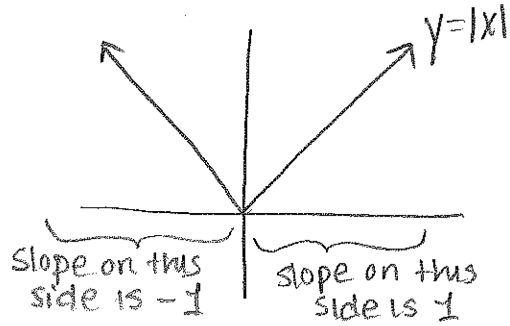
- $f(x)=C$ (a constant)
  slope everywhere is zero!
  $f'(x)=0$



- $f(x)=mx+b$ (straight line)
  Slope of tangent line at an point is m!
  $f'(x)=m$



Ex: $f(x)=|x|$



$$f'(x)=\begin{cases}-1, & x<0\\ 1, & x>0\end{cases}$$

What about at zero?

The issue: Go to def of tangent line

$$\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^-}\frac{|h|}{h}=-1$$

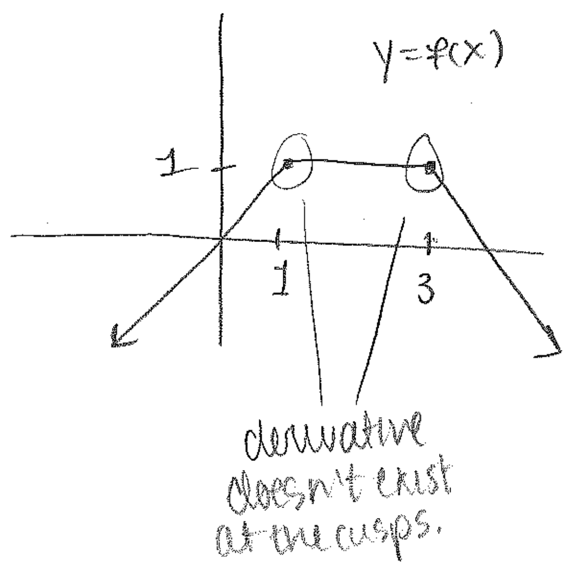
$$\lim_{h\to 0^+}\frac{|h|}{h}=1 \qquad \neq$$

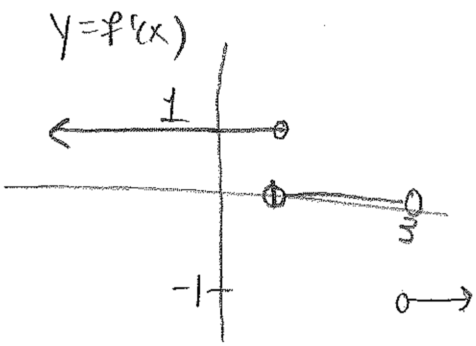
derivative doesn't exist when $x=0$. "cusp"

* We'll talk about graph shapes which cause problems a bit later *

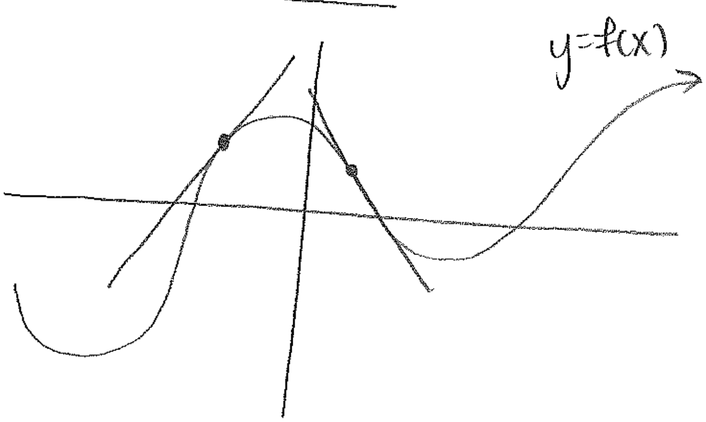
Ex: $f(x)=\begin{cases}x, & x\le 1\\ 1, & 1<x<3\\ -x+4, & 3\le x\end{cases}$

$$f'(x)=\begin{cases}1, & x<1\\ 0, & 1<x<3\\ -1, & 3<x\end{cases}$$





Graphs + Derivative



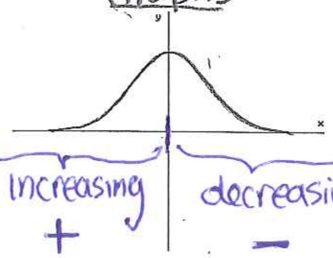
When $f(x)$ is increasing $f'(x)$ is positive

When $f(x)$ is decreasing $f'(x)$ is negative

Graph Match Up

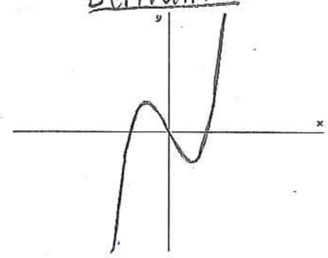
1. Fill in the Blank: Match the graphs at the left to the graphs of their derivatives on the right.

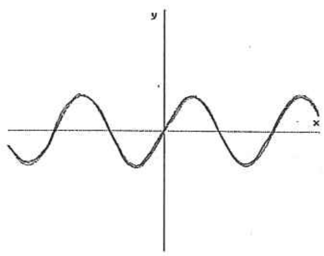
Graphs | Derivatives



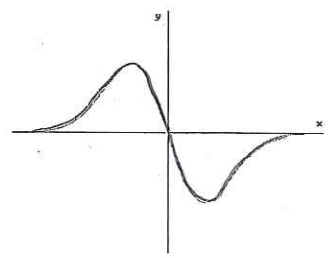
increasing + | decreasing −

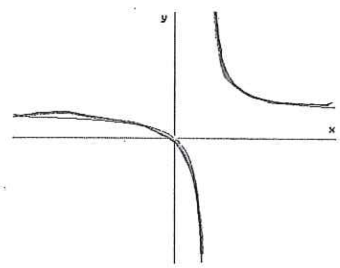
B

(A) 

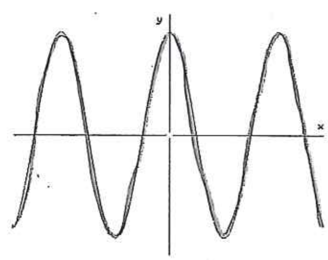


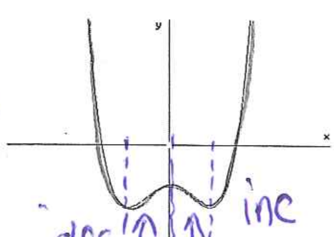
C

(B) 



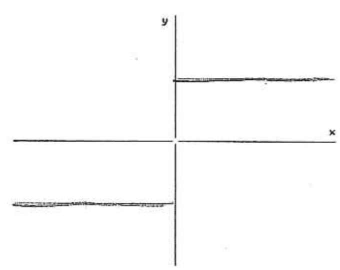
E

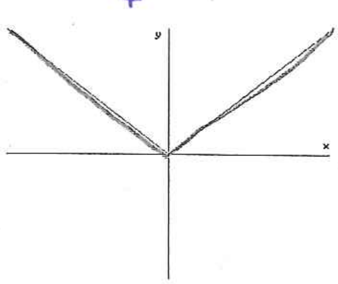
(C) 

☆ 

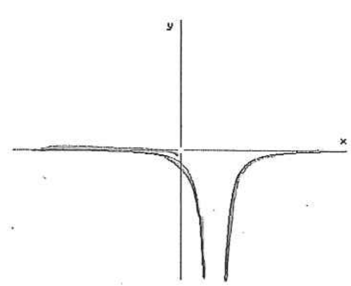
dec − | inc + | dec − | inc +

A

(D) 



D

(E) 

Go to work: building our first derivative rule.

Example 1: What *is* the derivative of x^2?

$$\begin{aligned}\frac{d}{dx}x^2 &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x)^2}{h}\\ &= \lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h} = \lim_{h\to 0}\frac{2xh+h^2}{h}\\ &= \lim_{h\to 0} 2x+h = \boxed{2x} \qquad \left(\text{so } \left.\frac{d}{dx}x^2\right|_{x=5} = 2*5\right)\end{aligned}$$

By taking limits, fill in the rest of the table:

$f(x)$	1	x	x^2	x^3	$\frac{1}{x}$	$\frac{1}{x^2}$	$\sqrt{x}$	$\sqrt[3]{x}$
$f'(x)$			$2x$					

Hints: For $\frac{1}{x^2}$, find a common denominator, and then expand.
For $\sqrt[3]{x}$, try multiplying and dividing by $(\sqrt[3]{x+h})^2 + (\sqrt[3]{x+h})(\sqrt[3]{x}) + (\sqrt[3]{x})^2$.

Do: Graph Match up

<u>More Notation</u>: Leibniz $f'(x) = \frac{d}{dx} f(x)$

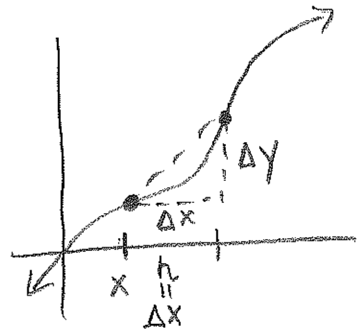
$$\frac{f(x+h) - f(x)}{h} = \text{is slope} = \frac{\text{rise}}{\text{run}} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

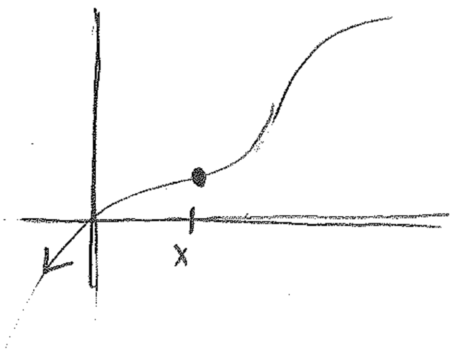
Δ means "change"

Leibniz did:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{dy}{dx} \sim \frac{\Delta y}{\Delta x}$$

(d instead of Δ when change is "infinitesimal")





So, we can write $f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)$

just another way to write derivative

<u>Derivatives at a pt:</u>

$f'(a)$ is der. of $f(x)$ evaluated at a.

Notation:

$$f'(a) = \frac{df}{dx}\bigg|_{x=a} = \frac{d}{dx} f(x)\bigg|_{x=a}$$

<u>Ex:</u> $\frac{d}{dx} x^2$ denotes der. of x^2

$\frac{d}{dx} x^2 \Big|_{x=5}$ is der. of x^2 at $x=5$

First derivative Rule:

(1) $$\frac{d}{dx}x^2 = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0} 2x+h = 2x$$

so $\frac{d}{dx}x^2 = 2x$

(2) $$\frac{d}{dx}x^3 = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$= \lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$= \lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}$$

$$= \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2$$

There is a rule...

If $f(x) = x^a$ then $f'(x) = \frac{d}{dx}x^a = a\cdot x^{a-1}$

(3) $$\frac{d}{dx}\sqrt{x} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Observe: $$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}(x)^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(\frac{1}{x^{1/2}}\right) = \frac{1}{2\sqrt{x}}$$

=

Works for negative powers too!

Let's take consecutive derivatives w/ power rule:

$$x^{5/2} \xrightarrow{d/dx} \frac{5}{2}x^{(5/2-1)} = \frac{5}{2}x^{3/2} \quad \left(1^{st} \text{ derivative} = f'(x) = \frac{d}{dx}x^{5/2}\right)$$

$$\xrightarrow{d/dx} \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)x^{3/2-1} = \frac{15}{4}x^{1/2} \quad \left(2^{nd} \text{ der} = f''(x) = \frac{d^2}{dx^2}x^{5/2}\right)$$

$$\xrightarrow{d/dx} \left(\frac{15}{4}\right)\left(\frac{1}{2}\right)x^{-1/2} = \frac{15}{8}x^{-1/2} = f'''(x) = \frac{d^3}{dx^3}x^{5/2}$$

$$\xrightarrow{d/dx} \frac{15}{8}\cdot\left(-\frac{1}{2}\right)x^{(-1/2-1)} = f^{(4)}(x) = \frac{d^4}{dx^4}x^{5/2}$$

$$= -\frac{15}{16}x^{-3/2}$$

DEF: (n^{th} derivative)

$$\underbrace{\frac{d}{dx}\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n} f(x) = \frac{d^n}{dx^n}f(x) = f^{(n)}(x)$$